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A nonlinear instability burst in plane parallel flow

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An infinitesimal centred disturbance is imposed on a fully developed plane Poiseuille flow at a Reynolds number R slightly greater than the critical value R_c for instability. After a long time, t, the disturbance consists of a modulated wave whose amplitude A is a slowly varying function of position and time. In an earlier paper (Stewartson & Stuart 1971) the parabolic differential equation satisfied by A for two-dimensional disturbances was found; the theory is here extended to three dimensions. Although the coefficients of the equation are complex, a start is made on elucidating the properties of its solutions by assuming that these coefficients are real. It is then found numerically and confirmed analytically that, for a finite value of $(R-R_c)t$, the amplitude A develops an infinite peak at the wave centre. The possible relevance of this work to the phenomenon of transition is discussed.

1. Introduction

Consider Poiseuille flow under pressure between two fixed parallel planes at a Reynolds number $R = U_0h/\nu$, where U_0 is the maximum velocity of the fluid, $2h$ is the distance between the planes and ν is the kinematic viscosity. We choose a set of Cartesian axes $Oxyz$ with origin O at some convenient point on the plane midway between the fixed planes, hx measuring distance in the direction of flow, hz distance in the direction perpendicular to the planes and hy distance in the spanwise direction. Then if $(U_0u,\ U_0v,\ U_0w)$ are the components of fluid velocity in the x, y, z directions respectively, the steady motion is given by

$$u = 1-z^2, \quad v = w = 0, \tag{1.1}$$

with the fixed planes at $z = \pm 1$.

In the classical theory of the stability of this flow to plane infinitesimal disturbances, the velocity components are taken to be

$$u = 1-z^2+\bar{A}\, e^{i\alpha(x-ct)}\,(d\psi_1/dz), \quad w = -i\alpha\bar{A}\, e^{i\alpha(x-ct)}\,\psi_1(z), \quad v = 0, \tag{1.2}$$

where ht/U_0 measures time, $\bar{A}$ is a constant whose square is negligible and α is given as a real wavenumber. The complex wave velocity c must be found, together with the eigenfunction $\psi_1(z)$ normalized so that $\psi_1(0) = 1$. On substitution into the Navier–Stokes equations, it is found that $\psi_1(z)$ must satisfy the fourth-order linear homogeneous Orr–Sommerfeld equation. The no-slip conditions at the fixed planes can be satisfied non-trivially only if c, α and R satisfy a certain functional relationship of the form

$$f(\alpha, c, R) = 0. \tag{1.3}$$

For R less than a critical value R_c it is known that $\operatorname{Im} c = c_i$ is negative for all real values of α, and it may be inferred that the basic flow is stable to infinitesimal (but not necessarily to finite) disturbances. The lowest value of R for which c_i vanishes is $R = R_c$ $(= 5774)$, with $\alpha = \alpha_c (= 1{\cdot}0202)$. For other values of α, $c_i < 0$ when $R = R_c$, while there is a range of values of α for which $c_i > 0$ in the case of $R > R_c$. In the neighbourhood of $R = R_c$ and $\alpha = \alpha_c$ the complex growth rate $-i\alpha c$, may be expanded as a double Taylor series:

$$-i\alpha c = -i\alpha_c c_{cr} + ia_{1r}(\alpha - \alpha_c) - a_2(\alpha-\alpha_c)^2 + (R - R_c)\, d_1 + \ldots, \tag{1.4}$$

where c_{cr}, a_{1r} are real constants and a_2, d_1 are complex constants with positive real parts. Their values, together with the values of R_c, α_c quoted above, have been kindly computed on behalf of the authors by Mr R. R. Cousins. He finds that

$$\left.\begin{aligned} c_{cr} &= 0{\cdot}2640, & a_{1r} &= -0{\cdot}384, \\ a_2 &= 0{\cdot}183 + 0{\cdot}070i, & d_1 &= (0{\cdot}17 + 0{\cdot}80i)\, 10^{-5}. \end{aligned}\right\} \tag{1.5}$$

The extension of this linearized theory to the study of the evolution of arbitrary infinitesimal disturbances has attracted much interest in recent years. The object of such investigations is to understand the phenomenon of transition, but we emphasize at once that there are formidable difficulties to be overcome before such studies can be applied to that phenomenon. One of these is that, for plane Poiseuille flow, transition can occur at Reynolds numbers of order $\frac{1}{5}R_c$; in Blasius flow, on the other hand, wave motions can occur for Reynolds numbers significantly lower than R_c, although the transition is not complete until R is of order $4R_c$. The value of such theoretical studies is clearly limited, therefore, but the understanding of the evolution of infinitesimal disturbances can be regarded as an essential first step towards the understanding of the evolution of the non-linear disturbances presumably present in transition. In order to derive the maximum benefit from such studies, however, it is important to make the basic assumptions as clear as possible and to insist that the theory is mathematically consistent. It should then be easier to assess the errors that arise if the theory is applied to situations of greater practical interest, when the basic assumptions of the theory are not strictly justified.

As a first step in such a theory, in a paper which we now refer to as I, Stewartson & Stuart (1971) suppose that at some instant of time the fluid is in steady motion with the velocity distribution defined by (1.1) and with $(R - R_c)$ small but positive. An arbitrary two-dimensional infinitesimal disturbance of characteristic amplitude Δ, centred on the plane $x = 0$, is then imposed on the flow. They define

$$\epsilon = (R-R_c)\,d_{1r}, \tag{1.6}$$

and assume that

$$\epsilon \ll 1, \quad \Delta \ll 1. \tag{1.7}$$

The growth of this disturbance is traced in I and it is shown that, when $R > R_c$, $t \gg 1$, $\epsilon t \ll 1$ and $|x+a_{1r}t| \ll |a_2 t|$, the velocity component w is proportional to

$$\mathrm{Re}\left\{\frac{-i\alpha_c\Delta}{t^{\frac{1}{2}}}E\exp\left\{\frac{\epsilon d_1 t}{d_{1r}}-\frac{(x+a_{1r}t)^2}{4a_2 t}\right\}\psi_1(z)\left[1+O\left(\frac{x+a_{1r}t}{a_2 t}\right)\right]\right\}, \tag{1.8}$$

where

$$E \equiv \exp\left[i\alpha_c(x-c_{cr}t)\right], \tag{1.9}$$

and u can be obtained from the continuity equation. Here $\psi_1(z)$ is the eigenfunction of the Orr–Sommerfeld equation associated with $\alpha = \alpha_c$, $R = R_c$. The form (1.8) is derivable from a synthesis of modes like (1.2).

When t becomes so large that ϵt is not small, the velocity field is not given by linear theory because of the exponential growth of w near $x+a_{1r}t = 0$. For this reason a self-consistent nonlinear theory is developed in I and may be summarized as follows. With scaled variables τ and ξ defined by

$$\tau = \epsilon t, \quad \xi = \epsilon^{\frac{1}{2}}(x+a_{1r}t), \tag{1.10}$$

an expansion of w in powers of $\epsilon^{\frac{1}{2}}$ of the form

$$w = -i\alpha_c\epsilon^{\frac{1}{2}}A(\tau,\xi)\,E\psi_1(z)+O(\epsilon) \tag{1.11}$$

is consistent with the Navier–Stokes equations only if

$$\frac{\partial A}{\partial \tau}-a_2\frac{\partial^2 A}{\partial \xi^2} = \frac{d_1}{d_{1r}}A+kA|A|^2. \tag{1.12}$$

The same equation has been derived by Watanabe (1969) for nonlinear instability waves in the one-dimensional flow of a plasma. However the derivation is simpler there, because there is no shear in the basic state corresponding to (1.1). Moreover, there is the difference that in his paper a_2, d_1 and k are all imaginary. In the context of fluid mechanics DiPrima, Eckhaus & Segel (1971) have obtained (1.12) for a more general class of flows, including plane Poiseuille flow, by a less direct and more complicated method. The properties of (1.12) are therefore seen to be central to the understanding of the nonlinear evolution of infinitesimal disturbances, and it is important to understand those properties fully, not only in the context of plane Poiseuille flow, but for general values of the constants.

With the definition $\psi_1(0) = 1$, so that $\epsilon^{\frac{1}{2}}AU_0$ is the amplitude of the velocity perturbation, the value of k at $\alpha = \alpha_c$, $R = R_c$ has been computed by Reynolds & Potter (1967) to be $\frac{3}{2}\alpha_c^2\,(19{\cdot}7-111i)$ when there is constant mass flux. (The factor $\frac{3}{2}\alpha_c^2$ allows for a difference in velocity scale.) A somewhat different value is given by Pekeris & Shkoller (1967, 1969); their calculation, however, is for constant pressure gradient, a requirement which is not justified for the disturbances we have in mind (see I, equation (3.14)). We also note that k changes rapidly with both α and R.

The final result stated in I is the initial condition satisfied by A as $\tau \to 0$. This follows from the terminal form (1.8) of the linearized theory by identifying it with (1.11) in the limit $\epsilon \to 0$, with $\tau \ll 1$ ($\tau = \epsilon t$). The error term $(x + a_{1r}t)/a_2 t$ in (1.8) is $O(\epsilon^{\frac{1}{2}}\xi/a_2\tau)$ when (1.10) is used, and we have

$$A \approx (\Delta/\tau^{\frac{1}{2}})\exp(-\xi^2/4a_2\tau) \quad \text{as} \quad \tau \to 0. \tag{1.13}$$

Thus the growth of A is centred on $\xi = 0$ as $\tau \to 0$; since we are concerned only with the evolution of a centred disturbance, we also require

$$|A| \to 0 \quad \text{as} \quad |\xi| \to \infty, \tag{1.14}$$

for $\tau > 0$. If this condition were violated, it would mean that A would be subject to the influence of disturbances emanating from, or present at, large distances. Furthermore, (1.13) may be contrasted with the work of Stuart (1960), who implicitly assumed an initial disturbance in the form of a normal mode with no spatial modulation. This is equivalent to taking A = constant at $\tau = 0$.

Our aim in the present paper is twofold. First, we wish to generalize (1.12) to treat arbitrary three-dimensional disturbances. This is achieved in §2 by the use of Squire's theorem (1933) relating the properties of three-dimensional perturbations to those of two-dimensional perturbations as given by solutions of the Orr–Sommerfeld equation and by use of a simple relation between (1.12) and (1.4). Second, in §§3–5 we begin a study of the properties of (1.12) for the case in which all the coefficients are real and $k > 0$. In §6 we make a similar study of the equivalent equation for three-dimensional perturbations and in §7 we discuss the solutions of the equation corresponding to (1.12) for subcritical flows. Finally, in §8 we comment on the possible relevance of the theory to the phenomenon of transition. Extensions of the theory to complex values of the coefficients have been made by Hocking & Stewartson (1971, 1972).

Since the equation (1.12) with real coefficients, but with $k < 0$, has already appeared in the studies by Newell & Whitehead (1969) and Segel (1969) of non-linear disturbances in Bénard convection, it is worth noting that the change in sign of k makes a crucial difference to the structure of the terminal solution. For example, let us take $a_2 = d_1 = 1$ and suppose $k = -1$. Then

$$\frac{\partial A}{\partial \tau} = \frac{\partial^2 A}{\partial \xi^2} + A - A^3, \tag{1.15}$$

and if A is a constant, A_0, at $\tau = 0$,

$$A = \frac{A_0 e^{\tau}}{[1 - A_0^2 + A_0^2 e^{2\tau}]^{\frac{1}{2}}} \quad \text{for} \quad \tau \geq 0, \tag{1.16}$$

so that $A \to 1$ as $\tau \to \infty$, which is the terminal solution. Another possibility for the terminal solution as $\tau \to \infty$ is that A tends to a function of ξ only. Such solutions have been exhibited numerically by Newell & Whitehead for bounded ξ and are periodic and nearly of square-wave form. No solutions in which A becomes infinite as τ increases have been found; moreover, it does not seem likely that solutions which satisfy the physical requirement of a centred disturbance, namely that $A \to 0$ as $|\xi| \to \infty$, exist.

If, on the other hand, we take $a_2 = d_1 = 1$ and $k = +1$ we have to solve

$$\frac{\partial A}{\partial \tau} = \frac{\partial^2 A}{\partial \xi^2} + A + A^3. \tag{1.17}$$

Application of the condition $A = A_0 = \text{constant}$ at $\tau = 0$ yields

$$A = \frac{A_0 e^{\tau}}{[1+A_0^2 - A_0^2 e^{2\tau}]^{\frac{1}{2}}} \quad \text{for} \quad \tau \geqslant 0, \tag{1.18}$$

which has the terminal solution $A \to \infty$ as $\tau \to \tau_0 = \frac{1}{2}\log(1+A_0^{-2})$. In this paper we find solutions of (1.17) when the initial condition is that A is a function of ξ; in most cases, we find that $A \to \infty$ as a certain finite value of τ is approached, but at one value of ξ only. The solutions have the character that as τ increases from zero, A spreads outwards as a function of ξ; moreover, after a brief initial fall, the maximum value of $|A|$ increases with τ. Eventually the cubic term in (1.17) dominates, causing an amplification of the rate of increase of the maximum value of $|A|$ and, for the relative magnitude, a reversal of the outward spreading in ξ. Indeed, a focusing phenomenon occurs and A approaches a delta function form near its maximum. The theory of course breaks down just before the delta function is achieved, specifically when $\epsilon^{\frac{1}{2}}A = O(1)$. In physical terms, the burst then occupies a length $O(h|\log\epsilon|^{\frac{1}{2}})$ of the channel. A similar situation occurs in the three-dimensional problem, in which a term $\partial^2 A/\partial\eta^2$ is added to the right-hand side of (1.17).

2. Three-dimensional disturbances

In order to develop the appropriate generalization of (1.12) to include three-dimensional disturbances, it is convenient to begin by reconsidering the relation between (1.12) and (1.4). If we neglect the nonlinear term in (1.12) the resulting linear equation has solutions of the form

$$A = e^{p\tau + i\gamma\xi}, \tag{2.1}$$

where γ is a real constant and

$$p = -a_2\gamma^2 + d_1/d_{1r}. \tag{2.2}$$

Hence the general linear form of $EA(\tau,\xi)$, in terms of x and t, is

$$\exp\{(p\epsilon + i\gamma a_{1r}\epsilon^{\frac{1}{2}} - i\alpha_c c_{cr})t + (i\gamma\epsilon^{\frac{1}{2}} + i\alpha_c)x\}, \tag{2.3}$$

which, when multiplied by $\psi_1(z)$, gives the form of w in (1.11). Furthermore, on making the identifications

$$\epsilon^{\frac{1}{2}}\gamma = \alpha - \alpha_c, \quad p\epsilon + i\gamma a_{1r}\epsilon^{\frac{1}{2}} - i\alpha_c c_{cr} = -i\alpha c$$

and using (1.6), we can equate (2.3) to the exponential assumed in the velocity structure (1.2), and also equate (2.2) to (1.4). Thus the linear terms in (1.12) could have been derived immediately from the relation (1.4), which is as it should be. For, any combination of solutions of the linearized disturbance equation with wavelengths approximately equal to $2\pi/\alpha_c$ can have its leading term put into the form

$$EA(\tau,\xi)\psi_1(z), \tag{2.4}$$

where $A(\tau,\xi)$ is a combination of terms like (2.1), and if we now ask what differential equation such an A must satisfy, we recover (1.12).

Alternatively, we may derive the form of the differential equation for A as follows. Let us assume that A, as a function of x and t, satisfies an equation of the general form

$$\frac{\partial A}{\partial t} = \theta_1 A - \theta_2 \frac{\partial A}{\partial x} - \theta_3 \frac{\partial^2 A}{\partial t^2} - \theta_4 \frac{\partial^2 A}{\partial x\, \partial t} - \theta_5 \frac{\partial^2 A}{\partial x^2} - \theta_6 \frac{\partial^3 A}{\partial t^3} - \dots, \tag{2.5}$$

where the θ_n are independent constants. If A is independent of x, corresponding to $\alpha = \alpha_c$, then θ_1 represents the complex growth rate and, with R slightly greater than R_c, θ_{1r} is small and positive ($\theta_{1r} = \mathrm{Re}\,\theta_1$). Furthermore, if A is given by (2.1) we have

$$\epsilon p = \theta_1 - \theta_2 i\gamma\epsilon^{\frac{1}{2}} = \theta_1 - \theta_2 i(\alpha - \alpha_c),$$

and since the growth rate, which is given by the real part of ϵp, is a maximum at $\alpha = \alpha_c$ it follows that θ_2 must be real. No other significant restrictions are available but if we now look for solutions in which A is a slowly varying function of t and x, it is natural to write $\theta_{1r} t = T$, $\theta_{1r} x = \bar{X}$. If we were to proceed to the limit $\theta_{1r} \to 0$, holding T and $\bar{X}$ finite, all but the first two terms of the right-hand side of (2.5) would vanish and we would be left with a first-order equation for A. This tells us that the important independent variable is $\bar{X} - \theta_2 T$, and on writing $(x - \theta_2 t)\,\theta_{1r}^{\frac{1}{2}} = X$ we reduce (2.5) to

$$\frac{\partial A}{\partial T} - \bar{a}_2 \frac{\partial^2 A}{\partial X^2} = A\left[1 + \frac{i\theta_{1i}}{\theta_{1r}}\right], \tag{2.6}$$

where $\bar{a}_2 = -\theta_5 + \theta_2\theta_4 - \theta_2^2\theta_3$ and the relative error is $O(\theta_{1r}^{\frac{1}{2}})$. The form (2.6) is not unique, since $\bar{X}$ can replace T as an independent variable, but the only change in (2.6) is that $\partial A/\partial T$ is replaced by $\theta_2 \partial A/\partial \bar{X}$ (compare with equation (5.1) of I). It is important to note that (2.6) cannot be extended; the higher order terms of (2.5) can only appear as small perturbations of (2.6). Having determined the form of the differential equation satisfied by A we can fix the values of the coefficients by a straightforward use of (1.4). This yields the linear terms in (1.12).

Let us next consider the nonlinear term. This arises from the inertia terms in the Navier–Stokes equations and involves products of derivatives with respect to x, z and t. However, A is a slowly varying function of x and t, and the leading nonlinear term must involve differentiations of E and $\psi_1(z)$ rather than A. Consequently the nonlinear term in (1.12) can contain no derivatives of A. Moreover, it must be identical in form with, and contain the same constant k as, the nonlinear term of the simple time-dependent equation (Stuart 1960).

It remains to derive the initial condition on A when $\tau \ll 1$ but $t \gg 1$. For this we neglect the nonlinear term of (1.12) and rewrite this equation in terms of t and X now defined to be $x + a_{1r} t$. We obtain

$$\frac{\partial A}{\partial t} - a_2 \frac{\partial^2 A}{\partial X^2} - \frac{\epsilon d_1}{d_{1r}} A = 0, \tag{2.7}$$

and look for solutions in which $A \to 0$ as $|X| \to \infty$ for fixed t. All such solutions are combinations of solutions of (2.7) of the form

$$A_n = t^{-\frac{1}{2}(n+1)} f_n(\gamma) \exp(\epsilon\, d_1 t/d_{1r}) \quad (\gamma = X(a_2 t)^{-\frac{1}{2}}), \tag{2.8}$$

where $n+1 > 0$, f_n is related to the parabolic cylinder functions and $f_n \to 0$ as $|\gamma| \to \infty$. In general, f_n is only algebraically small when $|\gamma|$ is large and consequently

$$t^{-\frac{1}{2}(n+1)} f_n(\gamma) = O(X^{-n-1}), \tag{2.9}$$

when $|\gamma| \gg 1$. Such a form is incompatible with our assumption of a centred infinitesimal disturbance at $t = 0$ for it suggests that the disturbed region is unlimited for all $t > 0$. These solutions correspond, we expect, to those resulting from the appearance of branch points in the Fourier–Laplace transforms referred to in I (p. 533). In order to avoid them we must choose n to be a positive integer or zero. Then the relevant solutions are linear combinations of

$$E^{\frac{1}{2}} \exp\left[\frac{\epsilon d_1 t}{d_{1r}} - \frac{X^2}{4a_2 t}\right], \tag{2.10}$$

together with its derivatives (of all orders) with respect to X and t. The exponential decay of these solutions as $|\gamma| \to \infty$ ensures that the disturbance is centred on $X = 0$, and we note that they correspond to the poles in the Fourier–Laplace transform in I. Consequently, when t is large A must be proportional to (2.10), and the appropriate initial condition for A in (1.12) is

$$A \approx \frac{\Delta}{\tau^{\frac{1}{2}}} \exp\left[\frac{d_1 \tau}{d_{1r}} - \frac{\xi^2}{4a_2 \tau}\right] \quad \text{as} \quad \tau \to 0, \tag{2.11}$$

where Δ is a representative amplitude of the disturbance at $t = 0$.

We are now ready to generalize (1.12) to include three-dimensional disturbances initially centred on O. An extended argument of the kind presented in I can be used to achieve this generalization, but here we obtain it by following the lines of argument set out above for the two-dimensional case. Both methods lead to the same fundamental equation.

Analogously to (1.2), we assume that as a result of a certain infinitesimal disturbance the velocity components are

$$\left.\begin{aligned} u &= 1 - z^2 + U_1(z) \exp[i\alpha x + i\beta y - i\alpha c t], \\ (v, w) &= (V_1(z), W_1(z)) \exp[i\alpha x + i\beta y - i\alpha c t], \end{aligned}\right\} \tag{2.12}$$

where U_1, V_1, W_1 are functions of z whose squares and products can be neglected. As before, the Navier–Stokes equations can only be satisfied by a solution of the form (2.12) if α, β, c and R satisfy a certain functional relationship which, as Squire (1933) showed, may be put into the form

$$f\{(\alpha^2+\beta^2)^{\frac{1}{2}}, c, \alpha R(\alpha^2+\beta^2)^{-\frac{1}{2}}\} = 0, \tag{2.13}$$

which is equivalent to the two-dimensional form (1.3) at a lower Reynolds number, but at the same value of c. It follows that, if ϵ is small, any value of $|\beta| \gg \epsilon^{\frac{1}{2}}$ will reduce the effective Reynolds number to a value less than R_c and the corresponding solution will be stable. Hence we must assume $\beta \sim \epsilon^{\frac{1}{2}}$, so that the functional form (1.4) generalizes to

$$\begin{aligned} -i\alpha c(1+\beta^2/2\alpha_c^2) = &-i\alpha_c c_{cr} + i a_{1r}(\alpha - \alpha_c + \beta^2/2\alpha_c) \\ &- a_2(\alpha - \alpha_c + \beta^2/2\alpha_c)^2 + d_1[R(1-\beta^2/2\alpha_c^2) - R_c] + \ldots, \end{aligned} \tag{2.14}$$

which can be written

$$-i\alpha c = -i\alpha_c c_{cr} + i a_{1r}(\alpha - \alpha_c) - a_2(\alpha-\alpha_c)^2 + d_1(R - R_c) - b_2\beta^2 + \ldots, \tag{2.15}$$

where

$$b_2 = \frac{d_1 R_c}{2\alpha_c^2} - \frac{i(a_{1r} + c_{cr})}{2\alpha_c} = 0{\cdot}0047 + 0{\cdot}081i, \tag{2.16}$$

using Cousins's numerical results. An argument entirely parallel to the earlier discussion of the two-dimensional disturbances shows that when $t \gg 1$ and $\epsilon t = \tau \ll 1$ the characteristic amplitude A of the velocity components must be a function of τ, ξ and η, where τ and ξ are defined by (1.10) and

$$\eta = \epsilon^{\frac{1}{2}} y. \tag{2.17}$$

Instead of (1.11) we now have, for the z component of velocity,

$$w = -i\alpha_c \epsilon^{\frac{1}{2}} E A(\tau, \xi, \eta)\, \psi_1(z) + O(\epsilon), \tag{2.18}$$

with corresponding expressions for the other velocity components. Equation (1.12) becomes

$$\frac{\partial A}{\partial \tau} - a_2 \frac{\partial^2 A}{\partial \xi^2} - b_2 \frac{\partial^2 A}{\partial \eta^2} - \frac{d_1}{d_{1r}} A = k|A|^2 A. \tag{2.19}$$

As in the two-dimensional case, there can be no derivatives in the nonlinear term, and (2.19) is the required generalization of (1.12).

It remains to derive the initial condition on A when $\tau \ll 1$ but $t \gg 1$. For this we neglect the right-hand side of (2.19) and rewrite it in terms of t, y and $X = x + a_{1r} t$. We obtain

$$\frac{\partial A}{\partial t} - a_2 \frac{\partial^2 A}{\partial X^2} - b_2 \frac{\partial^2 A}{\partial y^2} - \frac{\epsilon d_1}{d_{1r}} A = 0, \tag{2.20}$$

and look for solutions in which $A \to 0$ as $(X^2 + y^2) \to \infty$ for fixed t. All such solutions must be linear combinations of

$$t^{-1} \exp\left\{\epsilon \frac{d_1 t}{d_{1r}} - \frac{X^2}{4a_2 t} - \frac{y^2}{4b_2 t}\right\} \tag{2.21}$$

and its derivatives, of all orders, with respect to X and y. When t is large, the leading term in A must be proportional to (2.21) and the appropriate initial condition for A in (2.19) is

$$A \approx \frac{\Delta \epsilon^{\frac{1}{2}}}{\tau} \exp\left\{\frac{d_1 \tau}{d_{1r}} - \frac{\xi^2}{4a_2 \tau} - \frac{\eta^2}{4b_2 \tau}\right\}, \tag{2.22}$$

where Δ is a representative amplitude of the initial disturbance. It is noted that the index of τ is reduced from $-\frac{1}{2}$ to -1 on changing from two- to three-dimensional disturbances. In fact, with τ small (2.22) can be written as the product of two terms like (1.13), one for the ξ variation and one for the η variation.

Another possible form of initial disturbance is an oblique wave system, which exhibits a preference for some direction in the x, y plane. Suppose, for example, that initially v is of the form

$$v = \Delta\, e^{i\alpha_c x} f_1(x + my, z), \tag{2.23}$$

where f_1 is an arbitrary function and m is a constant, with corresponding forms for $[u - (1 - z^2)]$ and w. Then, by similar arguments, the initial condition on (2.19) when $t \gg 1$ and $\tau \ll 1$ is

$$A \approx \frac{\Delta}{\tau^{\frac{1}{2}}} \exp\left\{\frac{d_1 \tau}{d_{1r}} - \frac{(\xi + m\eta)^2}{4\tau(a_2 + m^2 b_2)}\right\}. \tag{2.24}$$

Although this form of disturbance is artificial, it is worth examining in order to establish the consistency of our approach. Throughout this discussion we have required that $A \to 0$ as infinity is approached away from the centre of the convected disturbance, be it the point $x + a_{1r}t = y = 0$, as in (2.22), or the line $x + a_{1r}t + my = 0$, as in (2.24). In order to be consistent, (2.22) and (2.24) must exhibit this property and so $\mathrm{Re}\,(a_2 + m^2 b_2) > 0$ for all real m. Using the numerical values (1.5) and (2.16) we find that this condition is satisfied. The fact that $\mathrm{Re}\, b_2$ is only just positive is not a lucky accident but a reflexion of the slow variation of c_i with Reynolds number near $R = R_c$ when $\alpha = \alpha_c$.

It is interesting to compare (2.19) with the equation derived by Newell & Whitehead (1969) and by Segel (1969) for the corresponding problem in Bénard convection. In this problem there is no preferred direction except that imposed by the choice of disturbance, whereas in our problem the direction of the initial steady flow provides such a preference. Newell & Whitehead consider disturbances whose main variations occur in the x and z directions, so that x is their preferred direction. It then follows that the coefficients corresponding to c_{cr}, a_{1r}, a_{2i} and d_{1i} in (1.4) are all zero and the equivalent of Squire's theorem simply replaces α by $(\alpha^2 + \beta^2)^{\frac{1}{2}}$ and leaves αc and R unchanged, because there is no mean convection in the Bénard problem. Thus (1.4) takes the form

$$-i\alpha c = -a_{2r}[(\alpha^2 + \beta^2)^{\frac{1}{2}} - \alpha_c]^2 + d_{1r}(R - R_c) + \ldots, \tag{2.25}$$

which reduces to

$$-i\alpha c = -a_{2r}[(\alpha - \alpha_c) + (\beta^2/2\alpha_c)]^2 + d_{1r}(R - R_c) + \ldots \tag{2.26}$$

when $\alpha - \alpha_c$ and β are small. If the variation in the y direction is ignored, $\beta = 0$ and the analogue of (1.12) is

$$\frac{\partial A}{\partial \tau} - a_{2r}\frac{\partial^2 A}{\partial \xi^2} = A - \bar{k}|A|^2 A, \tag{2.27}$$

where $\bar{k}$ is a known constant. The only significant difference between (2.27) and (1.12) is that $\bar{k}$ is real and positive. When, however, the variation in the y direction is introduced, (2.26) shows that the appropriate scaling for y is

$$\bar{\eta} = \epsilon^{\frac{1}{4}} y \tag{2.28}$$

instead of (2.17). The method explained in this section then leads quite simply to the equation

$$\frac{\partial A}{\partial \tau} - a_{2r}\left[\frac{\partial}{\partial \xi} - \frac{i}{2\alpha_c}\frac{\partial^2}{\partial \bar{\eta}^2}\right]^2 A = A - \bar{k}|A|^2 A, \tag{2.29}$$

which is quite different from (2.19). Newell & Whitehead arrived at (2.29) by a more direct method.

In our problem, we could choose an initial infinitesimal disturbance like that considered by Newell & Whitehead, with some selected preferred direction; it would only be necessary to require f_1 in (2.23) to be a slowly varying function of $x - y/m$, in addition to its dependence on $x + my$, since this would allow for slow variations across the planes $x + my =$ constant. However, the relation (2.15) between c, α and β cannot in general be put into the form (2.26) and, indeed, does not seem to permit any further simplification. In consequence, the governing equation for A is still (2.19) and the effect of the slow variation in f_1 is reflected only in the initial condition on A. Thus the equation (2.29) for Bénard convection is seen to be a very special case, valid only for certain values of the coefficients in (2.15) and for situations when the equivalent of Squire's theorem has a specially simple form.

It will be recognized that our object in this section has been to derive the basic amplitude equation (2.19) by simpler means than have been normal in previous literature, for example in I. Although we believe that the analysis given here is significantly simpler than in I, it remains a fact that if higher order terms are required for w in (2.18) as detailed a method as that in I will still be necessary. Especially, we note that such an analysis might be needed in order to obtain the equation for the amplitude function $A_2(\tau, \xi)$ of I (equation 4.8). However, so long as our main interest is in the amplitude equation (2.19), this need not concern us.

3. The fundamental equation: some possible terminal solutions

We begin our study of the equation (2.15) satisfied by A by neglecting the variation of A with η and by taking all the coefficients to be real. Then, without loss of generality, the governing equation can be reduced to a canonical form

$$\frac{\partial A}{\partial \tau} = \frac{\partial^2 A}{\partial \xi^2} + A + A^3, \tag{3.1}$$

together with the boundary condition $A \to 0$ as $|\xi| \to \infty$ for all τ and the initial condition

$$A \approx (\Delta/\tau^{\frac{1}{2}})\, e^{-\xi^2/4\tau} \quad (\tau \ll 1), \tag{3.2}$$

where Δ is a small positive constant.

The form of the solution when A is independent of ξ has already been given in (1.18), and it becomes infinite at a finite value τ_0 of τ, with

$$A[2(\tau_0 - \tau)]^{\frac{1}{2}} \to 1 \quad \text{as} \quad \tau \to \tau_0^-. \tag{3.3}$$

Thus one possible limiting structure for A has a singularity for all ξ at a finite value of τ, but since the initial condition on A is ξ-dependent, such a limiting form must allow for A to be a function of ξ near $\tau = \tau_0$. If we write

$$A[2(\tau_0 - \tau)]^{\frac{1}{2}} = 1 + B(\tau, \xi), \tag{3.4}$$

where $|B| \ll 1$ near $\tau = \tau_0$, the linearized equation for B is

$$\frac{\partial B}{\partial \tau} - \frac{\partial^2 B}{\partial \xi^2} = \frac{B}{\tau_0 - \tau} + B + 1$$

and we cannot find a solution (depending on both τ and ξ) which tends to zero as τ tends to τ_0. In addition, the limiting form (3.3) does not satisfy the boundary condition as $|\xi| \to \infty$, which makes it unlikely to be the correct limiting form of any solution of (3.1) which vanishes at infinity initially.

A second possibility for the limiting structure is that A tends to a function of ξ only, as $\tau \to \infty$. Such a limiting form exists but is periodic in ξ, with a period depending on the amplitude, and does not satisfy the condition $A \to 0$ as $|\xi| \to \infty$. Although such a failure is not necessarily lethal, since the limits $\xi \to \infty$, $\tau \to \infty$

have not been proved to be commutative, we can establish that A cannot change sign if it is initially positive. For the formal solution of (3.1) can be written

$$A = \frac{1}{2\pi^{\frac{1}{2}}}\int_0^\tau \frac{d\tau_1}{(\tau-\tau_1)^{\frac{1}{2}}}\int_{-\infty}^{\infty} d\xi_1 A^3(\tau_1,\xi_1)\exp\left[\tau-\tau_1-\frac{(\xi-\xi_1)^2}{4(\tau-\tau_1)}\right]+\frac{\Delta}{\tau^{\frac{1}{2}}}\exp\left(\tau-\frac{\xi^2}{4\tau}\right). \tag{3.5}$$

Suppose A first vanishes at $\xi = \xi_2$, when $\tau = \tau_2$. For all $\tau < \tau_2$ and finite ξ, $A > 0$ and, from (3.5), $A(\tau_2, \xi) > 0$ for all ξ, contradicting $A(\tau_2, \xi_2) = 0$.

Even when A changes sign initially, which might happen if for some reason the leading solution of the form (2.8) had a zero coefficient it is unlikely that A could become finitely oscillatory as $\tau \to \infty$. For in order to do so, an infinite number of zeros of A would have to be produced as τ increased to infinity, which would require minima of A, originally positive, to become negative. But when the value of A at a minimum is zero, $\partial A/\partial\tau > 0$, from (3.1), and A is increasing at that point and so cannot become negative. In a similar way, the alternative possibility that maxima of A which were originally negative could become positive can also be ruled out. The argument can easily be extended to cover the possibility that the second or higher derivatives of A with respect to ξ also vanish when a minimum or maximum value of A becomes zero.

A third possible limiting structure is for A to become singular at a finite value of τ but at a single value of ξ. The simplest way in which this might happen would be for A to take the form

$$\frac{1}{[2(\tau_0-\tau)]^{\frac{1}{2}}}g(\bar{\zeta}), \quad \bar{\zeta} = \frac{\xi-\xi_0}{(\tau_0-\tau)^{\frac{1}{2}}}, \tag{3.6}$$

where we are supposing that the singularity occurs at $\xi = \xi_0$ when $\tau = \tau_0$. Such a form achieves a balance between the three crucial terms of (3.1), namely $\partial A/\partial\tau$, $\partial^2 A/\partial\xi^2$ and A^3, and on substituting it into (3.1), we find that

$$g'' - \bar{\zeta}g' = g - g^3 \tag{3.7}$$

as $\tau \to \tau_0$ and that the boundary conditions are $g'(0) = g(\infty) = 0$. Although this equation looks hopeful, it has been established rigorously by Brown, in the appendix to this paper, that all solutions of (3.7) with a local maximum at $\bar{\zeta} = 0$ must become negative at some value of $\bar{\zeta} > 0$. Since we have already shown that $A > 0$ for all finite ξ it follows that there are no acceptable solutions of (3.7).

However, the forms (3.3) and (3.6) together contain the essentials of the appropriate structure, although as a historical fact it needed a study of one numerical solution of (3.1) to provide the key and to give the necessary confidence to unlock the secret. The difficulty is that (3.3), which corresponds to taking $g(\bar{\zeta}) = 1$, makes $A[2(\tau_0-\tau)]^{\frac{1}{2}} \to 1$ as $|\xi| \to \infty$, while, if $g(0) > 1$, A vanishes at a finite value of $\bar{\zeta}$, and if $g(0) < 1$, A has a minimum at $\bar{\zeta} = 0$. Thus $g(0) \leqslant 1$ is too small, but any $g(0) > 1$ is too big! The way of escape from this difficulty is to refine the definition of $\bar{\zeta}$ in (3.6) by introducing an appropriate power of $\log(\tau_0-\tau)^{-1}$. In the next section we set up a consistent expansion of A in the neighbourhood of $\tau = \tau_0$ and $\xi = \xi_0$.

4. The instability burst

We assume that A becomes singular at $\xi = \xi_0$ when $\tau = \tau_0$ and define

$$(2s)^{-\frac{1}{2}}\bar{B}(s,\bar{\zeta}) = A(\tau,\xi), \tag{4.1}$$

where $$s = \tau_0 - \tau, \quad \bar{\zeta} = p(\xi - \xi_0)\,s^{-\frac{1}{2}}$$

and p is a constant to be found. Then $\bar{B}$ satisfies

$$2s\frac{\partial \bar{B}}{\partial s} - \bar{\zeta}\frac{\partial \bar{B}}{\partial \bar{\zeta}} + 2p^2\frac{\partial^2 \bar{B}}{\partial \bar{\zeta}^2} + (2s-1)\,\bar{B} + \bar{B}^3 = 0. \tag{4.2}$$

It is surmised that $\bar{B}(s,0) \to 1$ as $s \to 0$, but we must not assume that, when s is small, $\bar{B}$ is a function of $\bar{\zeta}$ only, for this leads us back to (3.7). We introduce logarithms of s and write

$$s = e^{-\sigma}, \quad \zeta = \bar{\zeta}\sigma^{-\frac{1}{2}} = \frac{p(\xi - \xi_0)}{[(\tau_0 - \tau)\log(\tau_0 - \tau)^{-1}]^{\frac{1}{2}}}, \quad B(\sigma,\zeta) = \bar{B}(s,\bar{\zeta}). \tag{4.3}$$

The equation for B is then

$$B + \zeta\frac{\partial B}{\partial \zeta} - B^3 = \frac{2p^2}{\sigma}\frac{\partial^2 B}{\partial \zeta^2} + \frac{\zeta}{\sigma}\frac{\partial B}{\partial \zeta} - 2\frac{\partial B}{\partial \sigma} + 2e^{-\sigma}B. \tag{4.4}$$

The power of σ in the definition of ζ was chosen to effect a balance between the first two terms on the right-hand side of (4.4). The leading term in the expansion of B in descending powers of σ is found by neglecting the right-hand side of (4.4), giving

$$B = [1 + \bar{p}^2\zeta^2]^{-\frac{1}{2}}, \tag{4.5}$$

where $\bar{p}$ is a constant, interchangeable with p. Without loss of generality, we take $\bar{p} = 1$; the value of p will then be fixed by the required consistency of the expansion of B.

The next step is to set up a formal expansion of B in descending powers of σ:

$$B = \sum_{n=0}^{\infty} \sigma^{-n} B_n(\zeta), \tag{4.6}$$

where $B_0 = (1+\zeta^2)^{-\frac{1}{2}}$, and, when (4.6) is substituted into (4.4), we obtain

$$B_n + \zeta B_n' - \frac{3}{1+\zeta^2}B_n = G_n(\zeta) \quad (n \geqslant 1), \tag{4.7}$$

where G_n is a functional of $B_0, B_1, \ldots, B_{n-1}$. Hence

$$B_n = \frac{\zeta^2}{(1+\zeta^2)^{\frac{3}{2}}}\int_0^{\zeta}\frac{(1+\zeta_1^2)^{\frac{3}{2}}}{\zeta_1^3}G_n(\zeta_1)\,d\zeta_1, \tag{4.8}$$

and it may easily be seen by induction that $B_n \to 0$ as $\zeta \to \infty$ for all n. It is also crucial, however, that B_n be an analytic function of ζ at $\zeta = 0$, since otherwise the expansion (4.6) is not uniformly valid at the most significant place. If the integrand in (4.8), when expanded in ascending powers of ζ_1, contains a term proportional to ζ_1^{-1}, B_n would have a term proportional to $\zeta^2\log\zeta$ and this would lead to a term in B_{n+1} proportional to $\log \zeta$, which would destroy the validity of the expansion. Hence

$$G_n''(0) + 3G_n(0) = 0. \tag{4.9}$$

A further deduction from (4.8) is that $B_n(0) = -\frac{1}{2}G_n(0)$.

The coefficient of σ^{-1}, when (4.6) is substituted into (4.4), gives

$$G_1 = 2p^2 B_0'' + \zeta B_0', \tag{4.10}$$

and the consistency equation (4.9) gives

$$6p^2 = 1. \tag{4.11}$$

The value of B_1 can then be found by integrating (4.8) and is

$$B_1(\zeta) = \frac{1}{6(1+\zeta^2)^{\frac{3}{2}}} - \frac{\zeta^2 \log(1+\zeta^2)}{2(1+\zeta^2)^{\frac{3}{2}}} + \frac{p_{10}\zeta^2}{(1+\zeta^2)^{\frac{3}{2}}}, \tag{4.12}$$

where p_{10} is a constant to be found. We have now completed the structure of the dominant part of B near $\tau = \tau_0$ and it is instructive to write it in the original variables:

$$A \approx \left[2(\tau_0 - \tau) + \frac{(\xi - \xi_0)^2}{3\log(\tau_0 - \tau)^{-1}}\right]^{-\frac{1}{2}}. \tag{4.13}$$

Comparison with both (3.3) and (3.6) shows how near each is to the correct description. We can describe the significance of (4.13) as follows. The spreading out of the initial disturbance, which is characteristic of diffusion equations, is dominant at small times when A^3 is negligible, but near $\xi = \xi_0$ this spreading is overtaken by a tendency to focus, when A^3 is significant. The increase in A near $\xi = \xi_0$ rapidly gathers pace and, at a finite time, the solution bursts into a singularity there.

The present theory, which is based on the leading term in an expansion in powers of $\epsilon^{\frac{1}{2}}A$, breaks down when $\epsilon^{\frac{1}{2}}A = O(1)$. Hence (4.13) is only valid up to a time $\tau = \tau_0 - O(\epsilon)$, and the extent of the region occupied by the burst is then $|\xi - \xi_0| = O(\epsilon^{\frac{1}{2}}|\log \epsilon|^{\frac{1}{2}})$. In physical terms this means that, if the burst occurs at a time t_0, (4.13) holds up to a time $t_0 - O(h/U_0)$, and that the region where large values of A can be found then occupies a length $O(h|\log \epsilon|^{\frac{1}{2}})$ of the channel.

All is not quite well, however, with the expansion (4.6). On attempting to determine B_2 we find that

$$G_2 = \tfrac{1}{3}B_1'' + \zeta B_1' + 2B_1 + 3B_0 B_1^2, \tag{4.14}$$

so that

$$G_2(0) = \tfrac{2}{3}p_{10} + \tfrac{1}{4}, \quad G_2''(0) = -2p_{10} - \tfrac{49}{12}, \tag{4.15}$$

and (4.9) is violated whatever p_{10} is chosen.

The remedy is fortunately simple, at least in principle, and is well known in the literature of asymptotic expansions. The assumed form (4.6) is incomplete and must also include powers of $\log \sigma$. The leading terms can be written as

$$B = \frac{1}{(1+\zeta^2)^{\frac{1}{2}}} + \frac{1}{\sigma}\left[\frac{1}{6(1+\zeta^2)^{\frac{3}{2}}} - \frac{\zeta^2 \log(1+\zeta^2)}{2(1+\zeta^2)^{\frac{3}{2}}} + \frac{p_{10}\zeta^2}{(1+\zeta^2)^{\frac{3}{2}}}\right]$$
$$+ \frac{\zeta^2 p_{11} \log \sigma}{\sigma(1+\zeta^2)^{\frac{3}{2}}} + \frac{\log^2 \sigma}{\sigma^2} D_2(\zeta) + \frac{\log \sigma}{\sigma^2} C_2(\zeta) + \frac{1}{\sigma^2} B_2(\zeta) + o(\sigma^{-2}). \tag{4.16}$$

The equation for D_2 is the same as (4.7) with G_n replaced by $3p_{10}^2\zeta^4(1+\zeta^2)^{-\frac{7}{2}}$, so that

$$D_2 = -\frac{3p_{11}^2\zeta^2}{2(1+\zeta^2)^{\frac{5}{2}}} + \frac{p_{22}\zeta^2}{(1+\zeta^2)^{\frac{3}{2}}}. \tag{4.17}$$

The equation for C_2 is also the same as (4.7), but with a complicated right-hand side depending on the first three terms of the expansion (4.16). We shall not write it down, but note that the consistency requirement (4.9) is met for all p_{11}; this is not fortuitous, but a direct consequence of the fact that it is *not* met by (4.15). From this equation we deduce that

$$C_2(0) = -\tfrac{1}{3}p_{11}. \tag{4.18}$$

The equation for B_2 is now modified by the addition of a term $-2p_{11}\zeta^2(1+\zeta^2)^{-\frac{3}{2}}$ to the right-hand side of (4.14). It follows from (4.18) and (4.15) that

$$G_2(0) = \tfrac{2}{3}p_{10} + \tfrac{1}{4}, \quad G_2''(0) = -2p_{10} - \tfrac{49}{12} - 4p_{11} \tag{4.19}$$

and the consistency equation (4.9) is satisfied if

$$p_{11} = -\tfrac{5}{6}. \tag{4.20}$$

The expansion may now be continued indefinitely. Presumably the constants associated with D_2 and C_2 are determined from the terms of order σ^{-3} in the expansion and we should also get a relation connecting the constants p_{10} and p_{20} associated with B_2. If a similar difficulty to that which arose with B_2 in (4.15) should appear, it would only be necessary to add more terms involving appropriate powers of $\log \sigma$. The same remarks apply at each stage of the expansion. There must be an infinite number of arbitrary constants in the expansion, of course, but it is an open question whether they all occur in algebraic terms, or whether there is a finite number (possibly only one) associated with each of the infinite number of exponential terms from (4.4), each of which in turn is multiplied by an algebraic series like (4.16).

For a comparison with the numerical integration described in the next section the asymptotic expansion of $A(\tau, \xi_0)$ is useful. The terms found in this section give

$$A(\tau, \xi_0) \approx \frac{1}{[2(\tau_0-\tau)]^{\frac{1}{2}}}\left[1 + \frac{1}{6\sigma} + \frac{5\log\sigma}{18\sigma^2} - \frac{8p_{10}+3}{24\sigma^2} + \dots\right], \tag{4.21}$$

where p_{10} is an arbitrary constant.

5. Numerical investigations

The numerical work was undertaken in order to provide clues to the structure of the solution of (3.1) and to confirm, if possible, the analytical form of the asymptotic behaviour near the instability burst. The problem was to solve

$$\frac{\partial A}{\partial \tau} = \frac{\partial^2 A}{\partial \xi^2} + A + A^3, \tag{5.1}$$

with conditions

$$\partial A/\partial \xi = 0 \quad \text{at} \quad \xi = 0, \tag{5.2}$$

$$A \to 0 \quad \text{as} \quad \xi \to \infty, \tag{5.3}$$

$$A \approx (\Delta/\tau^{\frac{1}{2}})\, e^{-\xi^2/4\tau} \quad \text{for} \quad \tau \ll 1. \tag{5.4}$$

Two different finite-difference methods were used. If A_j and $\bar{A}_j$ denote the values of A at $\xi = jh$, at times τ and $\tau - \kappa$, respectively, the nonlinear term can be written

$$A_j^3 = \bar{A}_j^3 + 3\bar{A}_j^2(A_j - \bar{A}_j) + O(A_j - \bar{A}_j)^2. \tag{5.5}$$

Using backward differences for the time derivative and neglecting terms $O(\kappa^2)$, we obtain the finite-difference form of (5.1):

$$-A_{j-1} + \{2 + \kappa^{-1}h^2 - h^2(1 + 3\bar{A}_j^2)\}A_j - A_{j+1} = h^2\bar{A}_j(\kappa^{-1} - 2\bar{A}_j^2). \tag{5.6}$$

The second method was to use backward differences for the time derivative, but to use an iterative scheme to deal with the nonlinearity. If $A_i^{(n)}$ denotes the nth iterate of A_i, the equation is

$$-A_{j-1}^{(n)} + (2 + \kappa^{-1}h^2)A_j^{(n)} - A_{j+1}^{(n)} = h^2\{\kappa^{-1}\bar{A}_j + A_j^{(n-1)} + [A_j^{(n-1)}]^3\}, \tag{5.7}$$

with starting values at each step given by $A_j^{(0)} = \bar{A}_j$. The two methods were of comparable efficiency. The advantage of the matrix in (5.7) remaining the same, except when the step lengths were changed, was balanced by the necessity to perform several iterations for each time step.

The value of h, the step length in ξ, was kept fixed during each calculation. Initial trials suggested that $h = 0{\cdot}05$ would be sufficiently small, but later calculations were done with $h = 0{\cdot}025$. The value of κ, the step length in τ, was reduced whenever the maximum value of $A^2\kappa$ exceeded some small number, usually $0{\cdot}02$. This condition, which was suggested by the balance between the terms $\partial A/\partial\tau$ and A^3, was necessary in order to keep the relative change in A at each step small. Too large a value of κ could result in the singular behaviour being completely missed.

The boundary condition at $\xi = 0$ was incorporated into the set of equations by using the equation at $\xi = 0$, with $A_{-1} = A_1$. The outer boundary condition was replaced by the condition $A_N = 0$ at $\xi = Nh$. A rough guide to where the outer boundary could safely be placed was afforded by the solution of the linear equation, that is, (5.1) with the term A^3 missing. At large distances from the centre, where A is small, the linear solution can be expected to hold and it indicates that, as τ increases, the outer boundary should move outwards so that $(Nh)^2/4\tau$ remains constant and large. Values of Nh up to about 10 were usual, since the solution was only required for the limited time up to the appearance of the singularity. Instead of using an outer boundary moving in a predetermined way, another method was to start with the outer boundary at $Nh = 10$ and to increase the value of Nh by 1 whenever the value of A at the point next to the boundary exceeded 10^{-5}.

In the first set of calculations the initial values of A_j were all zero, except for A_0, which was set equal to 1, which is roughly equivalent to using (5.4) as the initial condition at $\tau = 10^{-4}$ with $\Delta = 10^{-2}$. The calculation produced a singularity of the predicted form at $\tau = 4{\cdot}23$. In order to avoid a great deal of uninteresting calculation, during which A gradually built up to values near 1, the initial part of the calculation was shortened by using the linear solution

$$A = (\Delta/\tau^{\frac{1}{2}})\,e^{\tau - \xi^2/4\tau} \tag{5.8}$$

to determine the starting values of A_j. The exact form used was

$$A = \tfrac{1}{2}e^{-\frac{1}{8}\xi^2}, \tag{5.9}$$

which corresponds to starting the calculation at $\tau = 2$ with $\Delta = 0{\cdot}095$. The maximum size of the neglected term A^3 was then only one-quarter of the terms retained and, since similar results were obtained for the form of the solution near the singularity from the two initial values, it was thought that the second form was sufficiently accurate. The singularity occurred at $\tau = 2{\cdot}947$, its earlier appearance being presumably associated with the larger value of Δ.

The results of the calculations revealed the behaviour described in §4, and are exhibited graphically in figures 1–4. The growth of A at the two stations $\xi = 0$ and $\xi = 1$ is plotted in figure 1, which shows how the comparatively slow

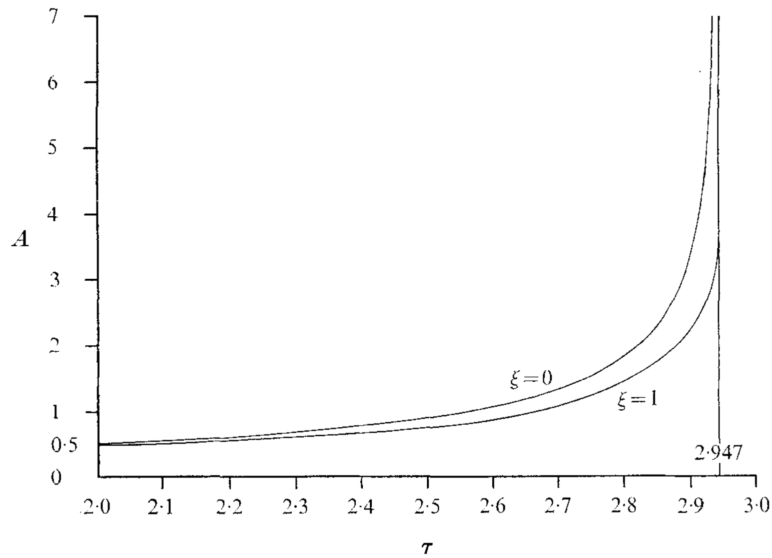

FIGURE 1. The values of A at $\xi = 0$ and at $\xi = 1$. Initial values given by (5.9).

increase in $A(\tau, 0)$ is followed by a very rapid rise as the singularity at $\tau = 2{\cdot}947$ is approached, whereas $A(\tau, 1)$ has a much more modest growth, in accordance with (4.13). The variation of A with ξ at various values of τ is shown in figure 2. The calculation started with values given by (5.9) at $\tau = 2$, and the curves for smaller values of τ were obtained from the linear solution. The initial drop and flattening of the distribution is followed by an exponential rise everywhere. The rise accelerates when the value of A exceeds 1, and the central part of the distribution then bursts away at a continually increasing rate.

A difficulty in comparing the numerical results with the theory is that the value of τ_0 is not accurately known. To avoid this difficulty $\tau_0 - \tau$ can be replaced by its value in terms of $A_0(\tau) = A(\tau, 0)$. Thus (4.13) can be written

$$A(\tau, \xi) \approx A_0\left[1 + \frac{A_0^2\xi^2}{3\log(2A_0^2)}\right]^{-\frac{1}{2}}. \tag{5.10}$$

Figure 3 shows $A(\tau, \xi)/A_0(\tau)$ plotted against $\zeta = A_0\xi[\log 2A_0^2]^{-\frac{1}{2}}$, together with the theoretical value $(1 + \frac{1}{3}\zeta^2)^{-\frac{1}{2}}$, at some of the values of τ used in figure 2. At the centre the values agree closely with the theoretical curve, and further out there is a steady approach to the limiting solution.

Similarly, the analytic expression (4.21) for the value of $A_0(\tau)$ can be written

$$\frac{1}{2A_0^2} \approx (\tau_0-\tau)\left[1-\frac{1}{3\log(2A_0^2)}-\frac{5\log\log(2A_0^2)}{9(\log 2A_0^2)^2}+\frac{2p_{10}+1}{3(\log 2A_0^2)^2}+\ldots\right]. \quad (5.11)$$

The computed values of $(2A_0^2)^{-1}$ plotted against τ are shown in figure 4 and reveal a nearly linear behaviour. The slow variation of the logarithms makes it difficult to confirm the other terms in (5.11). With $A_0 = 10$ the ratio of the third term to the second is about 0·5, and this ratio is only reduced to 0·4 when A_0 is as large

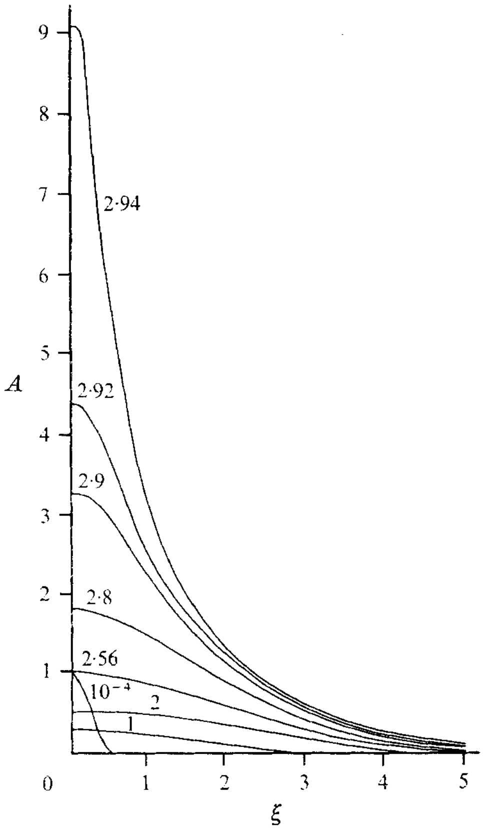

FIGURE 2. The changes in the shape of the variation of A with ξ as τ increases. The calculation began at $\tau = 2$ with (5.9) as initial values, and the curves for $\tau = 1$ and $\tau = 10^{-4}$ were obtained from the linear solution.

as 100. Detailed examination of the calculations showed that it was necessary to use extremely small step lengths, κ, when A_0 was large (for example, $\kappa = 10^{-4}$ when $A_0 \approx 7$), but also that, whenever the step length was changed, the differences did not fit smoothly together. The value of the coefficient $\frac{1}{3}(2p_{10}+1)$ of the last term in (5.11) was calculated for values of $\log 2A_0^2$ between 2 and 6, and, while it did not change as smoothly as was desired, it decreased from 1·7 to 1·4 as A_0 increased through this range, suggesting that p_{10} is approximately 1. In order to obtain more conclusive evidence of the accuracy of the terms in (5.11) it would be necessary to use a much smaller step length in ξ. It is considered, however, that the numerical work provides convincing evidence that the structure of the solution near the singularity has been obtained.

The initial condition (5.4) is the appropriate one to use, provided the coefficient Δ does not vanish. If $\Delta = 0$, the second term in (2.7) provides the

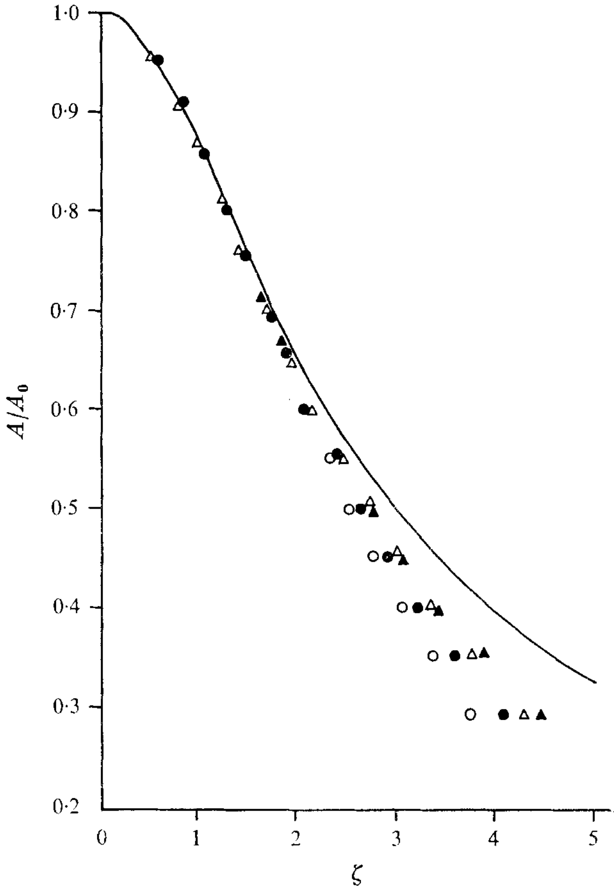

FIGURE 3. Comparison of the numerical solution with the theoretical value (5.10). ○, $\tau = 2{\cdot}8$; ●, $\tau = 2{\cdot}9$; △, $\tau = 2{\cdot}92$; ▲, $\tau = 2{\cdot}94$; ——, $(1+\frac{1}{3}\zeta^2)^{-\frac{1}{2}}$.

appropriate initial condition and can be found by differentiating (5.4) with respect to ξ, which gives

$$A \approx (\Delta_1/\tau^{\frac{3}{2}})\,\xi\, e^{-\xi^2/4\tau} \quad \text{for} \quad \tau \ll 1. \tag{5.12}$$

As long as the term A^3 is negligible,

$$A = (\Delta_1/\tau^{\frac{3}{2}})\,\xi\, e^{\tau-\xi^2/4\tau}, \tag{5.13}$$

which shows that the peak of the distribution is at the point $\xi = (2\tau)^{\frac{1}{2}}$ and that the maximum value increases like $\tau^{-1}e^{\tau}$, with a corresponding minimum at $\xi = -(2\tau)^{\frac{1}{2}}$. A limited amount of computation was done with this initial condition and the results agreed with the linear theory, until the maximum value became comparable with 1, when the nonlinear term became dominant and produced a singularity of the same kind as those obtained with the usual initial condition. During the rapid growth, the centre of the distribution was practically stationary. The initial condition in the calculation corresponded roughly to $\Delta_1 = 0{\cdot}003$ at $\tau = 0{\cdot}005$ and the singularity occurred at $\tau \approx 8{\cdot}7$.

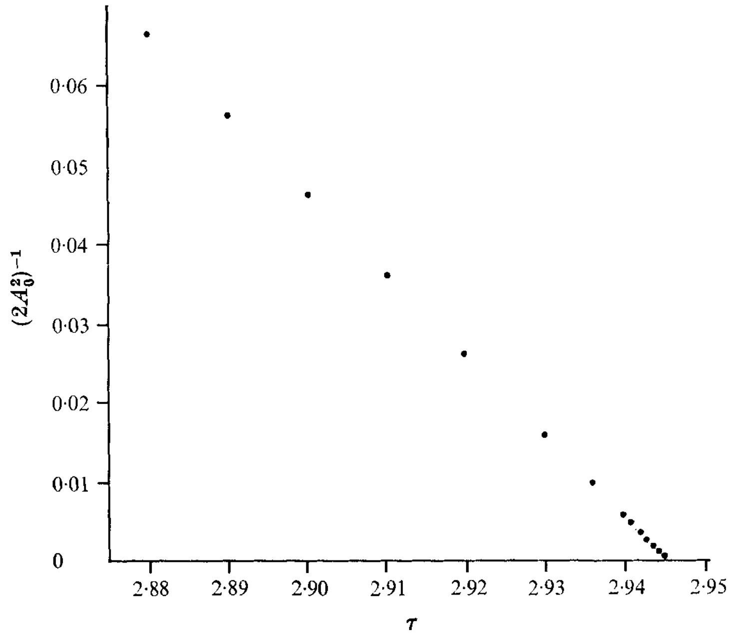

FIGURE 4. Values of $(2A_0^2)^{-1}$ *vs.* τ, showing the nearly linear behaviour in accordance with (5.11).

6. A three-dimensional burst

We now suppose that the original infinitesimal disturbance is centred at the origin, so that when the perturbations become nonlinear the amplitude function A satisfies (2.19) with the initial condition (2.22). We can no longer neglect the η variation, but, as before, we shall confine our attention to the simplest case, when all the coefficients in (2.19) are real. Without loss of generality we can reduce (2.19) to the form

$$\frac{\partial A}{\partial \tau} = \frac{\partial^2 A}{\partial \xi^2} + \frac{\partial^2 A}{\partial \eta^2} + A + A^3, \tag{6.1}$$

with the corresponding initial condition

$$A \approx \frac{\Delta_2}{\tau} \exp\left[-(\xi^2+\eta^2)/4\tau\right] \quad \text{as} \quad \tau \to 0, \tag{6.2}$$

where again $\Delta_2 = \Delta\epsilon^{\frac{1}{2}}$ is a small positive constant.

We shall now demonstrate that a limiting solution of (6.1) can be constructed, following the same lines as those successfully used in §4 for the two-dimensional burst. If A becomes singular at $\xi = \xi_0$, $\eta = \eta_0$, $\tau = \tau_0$, we define

$$B(\sigma, r, \theta) = [2(\tau_0 - \tau)]^{\frac{1}{2}} A(\tau, \xi, \eta), \tag{6.3}$$

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where

$$e^{-\sigma} = s = \tau_0 - \tau, \quad r = \left[\frac{(\xi-\xi_0)^2+(\eta-\eta_0)^2}{\sigma s}\right]^{\frac{1}{2}}, \quad \tan\theta = \frac{\eta-\eta_0}{\xi-\xi_0}. \tag{6.4}$$

Then B satisfies

$$B + r\frac{\partial B}{\partial r} - B^3 = \frac{2}{\sigma}\left(\frac{\partial^2 B}{\partial r^2} + \frac{\partial B}{r\,\partial r}\right) + \frac{2}{\sigma r^2}\frac{\partial^2 B}{\partial \theta^2} + \frac{r}{\sigma}\frac{\partial B}{\partial r} - 2\frac{\partial B}{\partial \sigma} + 2e^{-\sigma} B. \tag{6.5}$$

As in the two-dimensional problem, we now assume that B can be expanded in descending powers of σ in the neighbourhood of $\sigma = \infty$, but we shall expect that positive integral powers of $\log\sigma$ will also appear. The leading term of this expansion is obtained by neglecting the right-hand side of (6.5) and is

$$B_0 = [1+q(\theta)\, r^2]^{-\frac{1}{2}}, \tag{6.6}$$

which differs from the corresponding term (4.5) because q is a function of θ and not a constant.

The next terms in the expansion of B can be written

$$\frac{\log\sigma}{\sigma} B_{12}(r,\theta) + \frac{1}{\sigma} B_{11}(r,\theta), \tag{6.7}$$

and, from (6.5), B_{11} satisfies

$$B_{11} + r\frac{\partial B_{11}}{\partial r} - \frac{3}{1+qr^2} B_{11} = H_1(r,\theta), \tag{6.8}$$

where
$$H_1 = 2\frac{\partial^2 B_0}{\partial r^2} + \frac{2}{r}\frac{\partial B_0}{\partial r} + \frac{2}{r^2}\frac{\partial^2 B_0}{\partial \theta^2} + r\frac{\partial B_0}{\partial r}. \tag{6.9}$$

B_{12} satisfies an equation similar to (6.8), but with the right-hand side equal to zero. An equation analogous to(4.8) can be found which shows that the condition $B_{11} \to 0$ as $r \to \infty$ is automatically satisfied, but that B_{11} is analytic at $r = 0$ only if

$$(\partial^2 H_1/\partial r^2) + 3qH_1 = 0 \quad \text{when} \quad r = 0. \tag{6.10}$$

This consistency condition enables us to determine q. From (6.6) and (6.9) we have

$$H_1(0,\theta) = -4q - \frac{d^2 q}{d\theta^2}, \tag{6.11}$$

and
$$\frac{\partial^2 H_1}{\partial r^2} = 24q^2 - 2q + \frac{3}{2}\frac{d^2(q^2)}{d\theta^2} \quad \text{when} \quad r = 0. \tag{6.12}$$

Substitution into (6.10) gives

$$3\left(\frac{dq}{d\theta}\right)^2 + 12q^2 - 2q = 0, \tag{6.13}$$

which has solutions, either
$$q = \tfrac{1}{6}, \tag{6.14}$$
or
$$q = \tfrac{1}{6}\cos^2(\theta - \theta_0), \tag{6.15}$$
where θ_0 is an arbitrary constant.

The first solution clearly corresponds to a centred burst and can be expected to give the limiting form of A when the initial condition is (6.2), independent of θ. The form
$$(1+\tfrac{1}{6}r^2)^{-\frac{1}{2}}, \tag{6.16}$$
taken by B as $\sigma \to \infty$, is identical with the two-dimensional burst (4.5).

The second solution (6.15) is essentially two-dimensional in character since $r\cos(\theta-\theta_0)$ measures distance in a direction making an angle θ_0 with the ξ direction. The limiting form for B as $\sigma \to \infty$, namely

$$[1+\tfrac{1}{6}\{(\xi-\xi_0)\cos\theta_0+(\eta-\eta_0)\sin\theta_0\}^2/\sigma s]^{-\frac{1}{2}}, \tag{6.17}$$

is a generalization of (4.5) and can be expected to arise if the initial condition on A is derived from (2.24) instead of (6.2). As noted earlier, such a disturbance is probably artificial and of less interest than the point-centred disturbance, which leads to (6.16).

Further terms in the expansion of B can be worked out if desired, but we have not done so because no new points of interest seem to arise. Consequently the need for terms like B_{12} in the expansion has not been established. Finally, we note that if $6q = 1$

$$A(\tau,\xi_0,\eta_0) = [2(\tau_0-\tau)]^{-\frac{1}{2}}\{1+(1/3\sigma)+o(\sigma^{-1})\}, \tag{6.18}$$

since $B_{11}(0,\theta) = -\tfrac{1}{2}H_1(0,\theta)$.

7. Subcritical bursts

The theory described above is based on the assumption that the flow is slightly supercritical. It is easy to make a formal extension of the theory leading to (1.12) and (2.19) to cover subcritical flows, with $R < R_c$ and $|R-R_c|$ small. It is only necessary to replace ϵ by $|\epsilon|$, then the amplitude function $A(\tau,\xi)$ for two-dimensional disturbances satisfies

$$\frac{\partial A}{\partial \tau} - a_2\frac{\partial^2 A}{\partial \xi^2} = -A\frac{d_1}{d_{1r}} + k|A|^2 A \tag{7.1}$$

instead of (1.13), with the initial condition

$$A \approx (\Delta/\tau^{\frac{1}{2}})\, e^{-\xi^2/4a_2\tau} \quad \text{when} \quad \tau \ll 1. \tag{7.2}$$

However, Δ is a small positive constant and (7.1) tells us immediately that $A \to 0$ uniformly as $\tau \to \infty$, which simply means that the basic flow is stable to infinitesimal disturbances.

If we could choose a value for $|A| \approx 1$, it is clear that the nonlinear term in (7.1) could be destabilizing. On the basis of a self-consistent rational theory, such a value of Δ is excluded, for the initial condition is derived from the linear theory. During a time $t \sim 1$, nonlinear terms are assumed negligible and all wavelengths in an arbitrary infinitesimal disturbance are removed, leaving only those which are close to $2\pi/\alpha_c$. When t is large, but $|\epsilon|\,t$ small, we obtain (7.2) with $\Delta \ll 1$. Further, when $\Delta \approx 1$, (7.2) is inconsistent with (7.1), since the non-linear term is not negligible in comparison with the linear term when $\tau \ll 1$. We are therefore restricted to $\Delta \ll 1$ if we insist on a self-consistent theory.

Nevertheless the properties of the solution of (7.1) are very interesting, and it is likely that a study of this equation is worthwhile for the light which it may throw on subcritical instability and transition, which experimentally is known to occur (Davies & White 1928) at values of $R \approx \tfrac{1}{5}R_c$. Indeed, Reynolds & Potter (1967), who studied the simpler equation with $a_2 = 0$, hazarded that a turbulence level of only $2\frac{1}{2}\%$ is sufficient to provoke transition when $R = 1000$. Although we have been assuming throughout that R is close to R_c it should be noted that ϵ is a very slowly varying function of R at $\alpha = \alpha_c$; for example, when $R_c - R \approx 600$, $\epsilon = -10^{-3}$. Thus the requirement $|\epsilon| \ll 1$ may allow a larger range of R to be considered than might be expected. Also $|k| \approx 180$, so that even when $|A| = 0{\cdot}1$, the nonlinear effect can be significant.

One way by which an initial condition for (7.1) can be obtained is to take

$$A \sim \tau^{-\frac{1}{4}} F(\bar{\theta}) \quad \text{as} \quad \tau \to 0, \tag{7.3}$$

where $\bar{\theta} = \xi/(a_2\tau)^{\frac{1}{2}}$ and

$$F'' + \tfrac{1}{2}\bar{\theta}F' + \tfrac{1}{2}F + k|F|^2 F = 0. \tag{7.4}$$

Such a form is consistent with (7.1) and would agree with (7.2) if $F(0)$ were chosen to be small, provided $F'(0) = 0$. There seems no doubt that an F satisfying (7.4) and such that $F \to 0$ as $|\bar{\theta}| \to \infty$ does exist.

In discussing the possible limiting structures which (7.1) allows we shall abandon that part of the rational argument which fixed the initial condition as (7.2). Instead, we shall take the initial condition to be arbitrary, and write

$$A = A_1(\xi) \quad \text{at} \quad \tau = \tau_1. \tag{7.5}$$

The only restriction on A_1 is that we require $A_1 \to 0$ as $|\xi| \to \infty$. One possible limit is clearly $A \to 0$ as $\tau \to \infty$, and provided A_1 is sufficiently small, this limit will be achieved. In other words, the steady flow is stable to sufficiently small disturbances. In order to make further progress, we take a_2, d_1 and k to be real and defer the complex problem to another paper.

A second possible limit as $\tau \to \infty$ is $A \to A_2(\xi)$, where

$$A_2(\xi) = \left(\frac{2}{k}\right)^{\frac{1}{2}} \operatorname{sech} \frac{\xi}{a_2^{\frac{1}{2}}}, \tag{7.6}$$

which satisfies (7.1) with the time derivative absent. This limiting solution can, however, be shown to be unstable. We write

$$A = A_2(\xi) + A_\lambda(\xi)\, e^{\lambda\tau}, \tag{7.7}$$

and we must have $\operatorname{Re}\lambda < 0$ if $A \to A_2$, as $\tau \to \infty$. Substituting (7.7) in (7.1) we obtain the linear equation

$$a_2 A_\lambda'' - (1+\lambda) A_\lambda + 3kA_2^2 A_\lambda = 0, \tag{7.8}$$

and the eigensolutions A_λ must satisfy $A_\lambda \to 0$ as $|\xi| \to \infty$. One possible eigensolution is

$$A_\lambda = A_2^2 \tag{7.9}$$

and the corresponding eigenvalue is $\lambda = 3$. This positive eigenvalue establishes the instability of (7.6) as a possible limiting solution of (7.1).

A stable limiting solution of (7.1) can be expected to be the same as that discussed in § 4, since when A is large, the linear term on the right-hand side of (7.1) is insignificant. We conclude that with sufficiently weak initial disturbances the steady flow remains stable and the disturbances die out. On the other hand, if the initial disturbances are sufficiently large and $R_c - R$, although positive, is not too large, the disturbances grow indefinitely and terminate in an instability burst at a finite time.

These ideas about the structure of the limiting solutions of (7.1) have been tested numerically, using the methods described in §5, for the particular case $a_2 = d_1 = k = 1$. The initial values chosen were of the forms

$$A_1(\xi) = c_1 \exp(-c_2 \xi^2), \tag{7.10}$$

and

$$A_1(\xi) = c_1 \operatorname{sech}(c_2 \xi), \tag{7.11}$$

where c_1 and c_2 are constants. Both types of predicted behaviour were obtained. When the initial value of $\partial A/\partial \tau$ at $\xi = 0$ was positive, the value of A in the neighbourhood of that point began to increase, the increase continuing until an instability burst was produced. The numerical results suggest that, for initial disturbances centred at $\xi = 0$, a sufficient condition for an instability burst is

$$\partial^2 A_1/\partial \xi^2 - A_1 + A_1^3 > 0, \tag{7.12}$$

at $\xi = 0$. This condition is not necessary, since some solutions were obtained in which the central value of A initially decreased as the diffusion spread its effect outwards, but nevertheless A remained sufficiently large for the A^3 term to become dominant, resulting in an instability burst. Usually, however, an initial profile for which

$$\partial^2 A_1/\partial \xi^2 - A_1 + A_1^3 < 0 \quad \text{at} \quad \xi = 0 \tag{7.13}$$

resulted in the initial decrease in the value continuing unchecked and the whole disturbance dying out.

In figure 5, the values of A_1 and of $\partial^2 A_1/\partial \xi^2$ at $\xi = 0$ for the various calculations are shown. At the points labelled B the solution produced an instability burst, but at those labelled D the solution died away. It can be seen that the curve

$$\partial^2 A_1/\partial \xi^2 = A_1 - A_1^3 \quad \text{at} \quad \xi = 0 \tag{7.14}$$

provides a division between the two sets of points, except for the two points close to this boundary where the solution produced a burst, but the value of A initially decreased.

Point-centred disturbances to flows at subcritical Reynolds numbers can be discussed in a similar way. The governing equation for A becomes

$$\frac{\partial A}{\partial \tau} - a_2 \frac{\partial^2 A}{\partial \xi^2} - b_2 \frac{\partial^2 A}{\partial \eta^2} = -\frac{d_1}{d_{1r}} A + k|A|^2 A, \tag{7.15}$$

with initial condition

$$A \approx \frac{\Delta_2}{\tau} \exp\left\{ -\frac{\xi^2}{4a_2\tau} - \frac{\eta^2}{4b_2\tau} \right\}. \tag{7.16}$$

As before, $\Delta_2 \ll 1$ and we would like to relax this condition so that $A \sim 1$ when $\tau > 0$. Even if $\Delta_2 \ll 1$, (7.16) is not consistent with (7.15) for, at $\xi = \eta = 0$, the nonlinear term $\sim \Delta_2^3 \tau^{-3}$, while the linear terms $\sim \Delta_2 \tau^{-2}$. However, this non-uniformity can be overcome by supposing that the initial condition is imposed at a value of $\tau \sim \Delta_2^n$, where $\frac{1}{2} < n < 1$, or alternatively, by making a shift of the origin of τ on the left-hand side of (7.16) by an equivalent amount. Otherwise the situation is similar to the two-dimensional problem.

If all the coefficients in (7.15) are real, we can describe three possible limiting solutions. First, $A \to 0$ as $\tau \to \infty$, which is a stable solution and, provided A is sufficiently small at $\tau = 0$, will vanish in the limit $\tau \to \infty$. Second, there is a finite limit $k^{-\frac{1}{2}}A_\infty(r)$, where

$$A''_\infty + (1/r)\,A'_\infty - A_\infty + A^3_\infty = 0, \tag{7.17}$$

$r^2 = a_2^{-1}\xi^2 + b_2^{-1}\eta^2$ and $A_\infty \to 0$ as $\tau \to \infty$. Further, $A'_\infty(0) = 0$ to preserve A_∞ as a smooth function of ξ and η. No simple solution of (7.17) has been found, but a numerical solution has kindly been computed by Miss S. M. Burrough on behalf of the authors. With the additional constraint $A_\infty \geqslant 0$ suggested by the arguments in § 3 she finds that there appears to be a unique solution, for which $A_\infty(0) = 2{\cdot}206$.

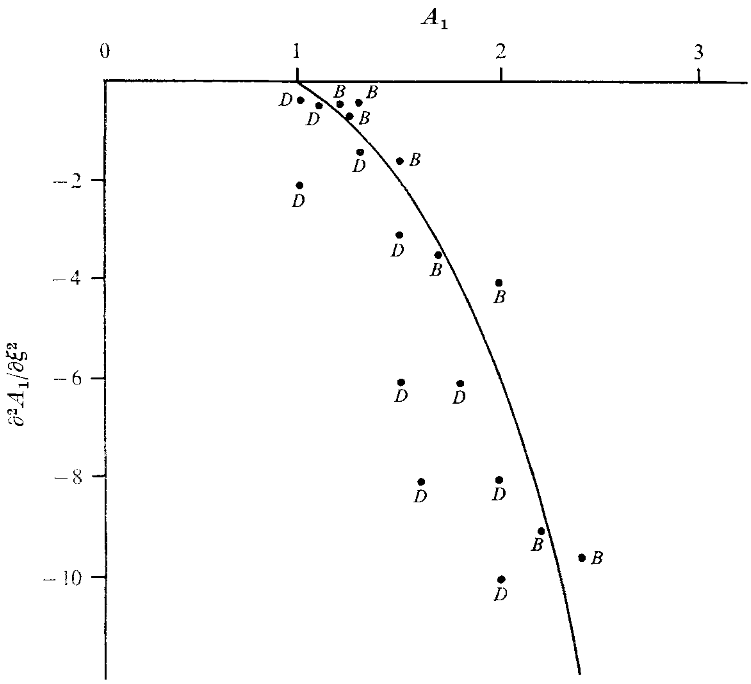

FIGURE 5. The values of A_1 and $\partial^2 A_1/\partial\xi^2$ at $\xi = 0$ used in the solution of (7.1) with $a_2 = d_1 = k = 1$. B denotes those values for which the solution showed a burst and D those for which the solution died away. Also shown is the curve (7.14).

A similar problem to (7.17) has been extensively studied in connexion with certain problems in elementary particle physics [see Anderson & Derrick (1970) for details and further references]. The only change is that the term $r^{-1}A'_\infty$ is replaced by $2r^{-1}A'_\infty$. It has been shown that there is an infinity of solutions of this equation and in all but the simplest A_∞ vanishes at least once in $r > 0$. Further, in the context of the nonlinear wave equation, the solutions are unstable, leading to bursts similar to those discussed here. It seems quite likely therefore that there is also an infinite number of solutions to (7.17) and that they are all unstable.

The third limiting solution can be expected to be the same as that discussed in § 6, for again the sign of the linear term on the right-hand side of (7.15) is insignificant when A is large. As in the case of the two-dimensional disturbance, we conclude that a point disturbance, which is sufficiently large initially, can grow when $R < R_c$ and then terminates in an instability burst, the last stages of which, when $\tau_0 - \tau = O(\epsilon)$, cannot be described by the present theory.

8. Discussion

The notion of a rational theory in fluid mechanics was explained by Van Dyke (1964) and may be restated in the following way. The region in which we are interested is divided into a number of domains and in each the solution is expanded in a consistent series of ascending powers of some small parameter ϵ. The various domains are also assumed to intersect in some sense and the expansions are matched in these intersections. Thus, so far as can be tested, the solution is fully consistent but, on the other hand, the theory is by no means rigorously established. It remains to prove that the series are asymptotic and that the domains in which they are valid do intersect. Nevertheless, we confidently claim that the results are correct and that a rigorous proof would establish the validity of the assumptions.

In the present problem the situation is complicated by the presence of two small parameters, Δ and ϵ, one representing the amplitude of the initial disturbance and the other the growth rate of the unstable mode according to linearized theory. Thus, strictly, a double asymptotic expansion is required; but it happens that it is only necessary to consider the leading term in the Δ expansion. In the first domain considered, defined by $t < \infty$, ϵ plays no special role and the development of the solution to order Δ is firmly based on linearized theory and the Orr–Sommerfeld equation. When t is large but ϵt is small, this theory predicts that a general initial disturbance will grow exponentially with growth rate ϵ. The second domain considered is $t \gg 1$, $\epsilon t = \tau \sim 1$, and here the theory is based on the nonlinear studies developed in earlier papers (Stuart 1960; Watson 1960*a*; Stewartson & Stuart 1971). In this domain the leading term of the perturbation is $O(\epsilon^{\frac{1}{2}})$ independently of Δ and the two solutions are matched in the intersecting part of the two domains, namely $t \gg 1$, $\epsilon t \ll 1$. The presence of Δ in the solution in the first domain and its disappearance from that in the second can be accounted for by an origin shift in $t \sim \log \Delta\epsilon$.

In contrast, much earlier work has considered *either* the linear Orr–Sommerfeld equation only, and therefore restricted attention to the first domain, *or* nonlinear developments involving interactions between one or more normal modes, which, while being very interesting and offering clues to the understanding of the transition problem, are not rational in the sense used here. Examples of linearized analysis are afforded by the studies of Benjamin (1961), Criminale & Kovasznay (1962) and Gaster (1968*a, b*) on the evolution of two- and three-dimensional wave systems, essentially (in our terminology) according to the linearized forms of (1.12) and (2.19). On the nonlinear side, the interesting studies by Benney and Lin (Benney & Lin 1960; Benney 1961, 1964; Lin & Benney 1966) on interacting three-dimensional modes are not rational, because they utilize an empirical assumption of the equivalence of the two wave speeds, which, after ten years, is still as far as ever from justification. We shall return to this point later. An exception to this general criticism is provided by the work of Stuart (1960) and

Watson (1960*a*). Their studies are rational in our interpretation, but concentrate on the nonlinear evolution of normal modes, so that the initial-value problem solved by them is very special indeed. We believe that the present description for the first time unifies the study of the evolution of small disturbances at Reynolds numbers just above the critical and provides a firm base for further development.

Having obtained a description of the nonlinear development of the disturbance in terms of the solution A of a partial differential equation, the rest of the paper has been concerned with a discussion of some of its main properties. Earlier, the properties of A when it is a function of τ only have been studied by Stuart (1958, 1960) and others, who introduced the notions of amplitude equilibrium and of a threshold amplitude requirement for instability. In this paper we have begun to extend these ideas by requiring A to be a function of position as $\tau \to 0$, but have added the simplifying assumption that all the coefficients of the differential equation are real. The principal new result which emerges is that of the explosive 'burst' of the nonlinear oscillations at some finite value τ_0 of $\tau(=\epsilon t)$ and at one point only, either of ξ in two dimensions or of ξ and η in three. The burst is characterized by a singularity in A and it should be borne in mind that the theory ceases to be rational at such an event, although it is formally valid up to $\tau = \tau_0 - O(\epsilon)$. If this phenomenon of focusing and bursting is characteristic of solutions of the differential equation even when the coefficients are not real,† then we may conclude that an initial small disturbance, which linear theory predicts will travel downstream with the group velocity, growing in amplitude and spreading outwards, will, through the agency of nonlinear effects, ultimately concentrate energy towards the centre of the disturbance. The amplitude will grow explosively until, in a region $O(h|\log \epsilon|^{\frac{1}{2}})$, i.e. of order just larger than the depth of the channel, the assumption of small disturbances fails and the theory ceases to be valid.

It is natural to wonder whether the present work has relevance to the study of instability and transition in parallel shear flows and turbulence. Of special interest is the question of the relevance of the self-focusing burst to the embryo turbulent spots noticed and recorded in some detail by Klebanoff, Tidstrom & Sargent (1962), Kovasznay, Komoda & Vasudeva (1962), Tani (1969) and Komoda (1967). In its explosiveness the phenomenon of the burst, as described in this paper, certainly bears a superficial resemblance to those observations of turbulent spots. It would be premature to claim more at this stage of the development of the theory, however. In the first place, the rational part of the theory is restricted to Reynolds numbers just above R_c and it must be conceded that a basic assumption of the theory, namely that the flow is initially fully developed and undisturbed, is unattainable in practice at such values of R. Second, spots have mainly been observed in boundary layers, at Reynolds numbers $R \sim 3R_c$ (Klebanoff *et al.* 1962); on the other hand, in plane Poiseuille flow transition occurs at $R \sim \frac{1}{5}R_c$ and the relevance of spots is not known. Third, the embryo spots (or breakdown) as described, for example, by Klebanoff *et al.* (1962) and illustrated by

† *Note added in proof.* A subsequent description of the properties of the solutions when the coefficients are complex has been given by Hocking & Stewartson (1971, 1972).

detailed figures in Komoda (1967), show a strong three-dimensional character. Our theory does permit a three-dimensional eruption (§ 6), but its exact character for boundary layers cannot be forecast yet. However, there are indications that its three-dimensional character is weak ($\beta \ll \alpha_c$). In the experiments quoted above $\beta \sim \alpha_c$, but the disturbances are forced by a vibrating ribbon with corresponding and imposed spanwise variations.

The weight of current descriptions of spot formation in experiments generally ascribes it to the 'nonlinear effect of a three-dimensional perturbation' (Klebanoff *et al.* 1962). [Their paper makes it clear that in certain regions known as 'peaks', local shear layers develop and shed 'spots' of high-frequency velocity fluctuations. Theories ascribing the spot to a local instability of the shear layer have been given by Greenspan & Benney (1963) and Stuart (1965).] However, the fact that three-dimensional and nonlinear effects occur together does not prove that these aspects are necessarily of equal importance for spot formation, although a number of empirical arguments are suggestive. The plain fact of the matter is that, in spite of a wealth of beautiful experimentation, no theory remotely adequate to explain them has so far been advanced. The most significant of such theories is perhaps that of Benney & Lin (1960), Benney (1961, 1964) and Lin & Benney (1966), which has often been favourably assessed (Klebanoff *et al.* 1962; Tani 1969; Mollo-Christensen 1971) by experimenters. In this theory it is found that the interaction of a two-dimensional wave (like (1.2)) with two three-dimensional waves (like (2.12), but with β taking two values equal in magnitude but opposite in sign) can lead to a longitudinal vortex system with greatest strength near the *convex* part of the Kelvin cat's eyes. However, as Stuart (1961, 1962) immediately pointed out, the theory makes the unjustified assumption that the corresponding values of c_r are equal for the two- and three-dimensional waves, whereas they are known to differ by a factor of order up to 15 % in Blasius flow. Thus the theory is not rational in Van Dyke's sense. For it to be so, it would have to be restricted to the neighbourhood of $R = R_c$ and to $\beta \ll 1$ in order to keep the phases of the waves the same. It would then reduce to a form equivalent to, and indeed almost identical with, the theory of the present paper. As it is, the assumption of synchronization is formally unjustified, since the interaction takes a time ϵ^{-1} to develop and the phase coherence on which the theory depends has then been completely lost. We further note that the claim of substantial agreement between their theory and the experiments of Klebanoff and others, can easily be destroyed if the theory is empirically modified to allow for this phase difference (Stuart 1961). Thus, although the Lin–Benney theory is interesting and suggestive, it cannot be regarded as explaining the phenomenon of longitudinal-vortex structure followed by embryo spots. In view of the limitations already mentioned, it would be premature to claim that the burst found here is necessarily relevant to embryo spots. Nevertheless, the theory shows for the first time that nonlinear effects can lead to focusing and, since it is self-consistent, the theory also provides a firm basis for further research.

Another aspect of nonlinearity which may be worth noting is related to the assumption of initially undisturbed and fully developed plane parallel flow in our theoretical work. In a boundary layer the spread of the basic flow leads to an increase in the Reynolds number with downstream distance. Consequently a wave which initially may be close to the critical Reynolds number propagates into regions where the Reynolds number is higher. However, in such regions a broader band of three-dimensional waves is possible, as the work of Watson (1960*b*) and Michael (1961) shows. If we couple this feature with the idea of 'resonance' of three-dimensional oscillations (Raetz 1959; Stuart 1962; Craik 1971) we have one possible mechanism by which three-dimensionality can become important in association with nonlinearity. The validity of such mechanisms remains to be assessed.

Finally we note that, just before the first submission of this paper, Dr M. Gaster informed one of us (J.T.S.) that, in an uncompleted independent investigation, he had extended his earlier work (1968*a*, *b*) and conceived amplitude equations similar to (1.12) and (2.19).

One of us (K.S.) is grateful to Prof. G. L. von Eschen of the Department of Aeronautical and Astronautical Engineering, The Ohio State University, for providing hospitality while the final revision of this paper was being made.

Appendix. On the differential equation $\ddot{x} = t\dot{x}+x-x^3$

By S. N. Brown, University College, London

In the main body of the paper the authors noted that a possible similarity solution to their partial differential equation (3.1) given by the ordinary differential equation (3.7) is unacceptable because the required boundary conditions cannot be satisfied. With a change of notation to facilitate comparison with kinematics, the equation is $\ddot{x} = t\dot{x}+x-x^3$, and the solution is to be such that $x(t)$ has a positive maximum at $t = 0$, and $x(t) \to 0$ monotonically as $t \to \infty$. This means $\dot{x}(0) = 0$ and $\ddot{x}(0) < 0$, and requires $x(0) = 1+\alpha$ $(\alpha > 0)$. The following theorem establishes, as stated by the authors, that with these initial conditions $x(t) = 0$ at a finite value of t.

Theorem. Suppose $x(t)$ is a real function of t satisfying

$$\ddot{x}(t) = t\dot{x}(t)+x(t)-x^3(t) \quad \text{when} \quad t \geqslant 0 \tag{A 1}$$

and

$$\dot{x}(0) = 0, \quad x(0) = 1+\alpha \quad (\alpha > 0). \tag{A 2}$$

Then $x(t) = 0$ for some t.

We think of the equation as determining the motion on the x axis of a particle P subject to two forces: one being an attraction to the point $x = 1$ and the other depending on its velocity. Initially P is at rest at $x = 1+\alpha$, its acceleration is $-\alpha(1+\alpha)(2+\alpha)$, and the acceleration remains negative and the speed of P increases as long as $x > 1$. P therefore reaches 1; suppose it does so for the first time when $t = t_1$ and write $\dot{x}(t_1) = -u_1$. If $\ddot{x}(t)$ remains negative when $t > t_1$ and $0 < x < 1$, the speed of P will always exceed u_1 and P will reach the origin. This will be shown to be the case.

We shall apply the following simple deductions from the mean-value theorem for the function $x(t)$.

(i) If $\ddot{x}(t) < 0$ for $0 < t < k$, then $x(t) - (1+\alpha) > t\dot{x}(t)$, and hence $\{1+\alpha - x(t)\}/t$ increases with t during this time.

(ii) If $\ddot{x}(t) > 0$ for $0 < t < k$, then $x(t) - (1+\alpha) > \frac{1}{2}t^2\ddot{x}(0)$ throughout $(0, k]$.

In particular, we deduce from these that

$$t_1 u_1 > \alpha, \tag{A 3}$$

that

$$\alpha t' < t_1\{1+\alpha - x(t')\} \tag{A 4}$$

provided that $\ddot{x}(t) < 0$ when $0 < t < t'$ and $t' > t_1$, and that

$$t_1 > \frac{2}{(1+\alpha)(2+\alpha)} \tag{A 5}$$

if $\ddot{x}(t) > 0$ throughout $(0, t_1)$.

The result to be proved is intuitively obvious for large α, for in this case the speed of arrival of P at 1 is so large that it cannot be reduced by the attractive force so as to prevent P from reaching the origin. It is in fact easy to prove the theorem when $\alpha > 0{\cdot}1$. To see this note that if $\ddot{x}(t) < 0$ throughout $(0, T)$, then by (A 1) and (i)

$$\ddot{x}(t) < -x^3 + 2x - 1 - \alpha \quad \text{in} \quad (0, T).$$

A simple calculation shows the maximum of $-x^3 + 2x - 1$ when $x > 0$ to be less than $0{\cdot}1$. Thus if $\alpha > 0{\cdot}1$, $\ddot{x}$ remains negative as long as $0 < x < 1$ and P reaches the origin.

For a general proof we have to consider $\dddot{x}$ and x^{iv}. We display the information about $x^{(r)}(t)$, (more especially at $t = 0$ and $t = t_1$) in table 1.

t ...	0	t_1	
x	$1+\alpha$	1	
$\dot{x}$	0	$-u_1$	
$\ddot{x} = t\dot{x} + x - x^3$	$-\alpha(1+\alpha)(2+\alpha)$	$-t_1 u_1$	(A 6)
$\dddot{x} = t\ddot{x} + (2-3x^2)\dot{x}$	0	$u_1(1-t_1^2)$	(A 7)
$x^{\text{iv}} = t\dddot{x} + 3(1-x^2)\ddot{x} - 6x\dot{x}^2$	$3\alpha^2(1+\alpha)(2+\alpha)^2$	$u_1\{t_1(1-t_1^2) - 6u_1\}$	(A 8)

TABLE 1

It will emerge that $t_1 < 1$, i.e. $\dddot{x}(t_1) > 0$; this makes it plausible that $\ddot{x}(t)$, negative at t_1, might become zero. However, it will be shown that this is offset by the negative values of $x^{\text{iv}}(t)$ when $t \geqslant t_1$.

We prove first that

$$0 < 1 - t_1^2 < \tfrac{3}{2}\alpha, \tag{A 9}$$

equivalent by (A 7) to

$$0 < \dddot{x}(t_1) < \tfrac{3}{2}\alpha u_1. \tag{A 10}$$

To prove that $\dddot{x}(t_1) > 0$, we note first that, since $\dddot{x}(0) = 0$ and $x^{\text{iv}}(0) > 0$, there is an interval, say $(0, \tau]$ with $\tau < t_1$, in which $\dddot{x}(t) > 0$. $t\ddot{x} - \dot{x}$ therefore increases in $(0, \tau]$ from 0 to some positive value p. As long as $\tau < t < t_1$ and $\dddot{x}(t) > 0$, we have $t\ddot{x} - \dot{x} > p$ and so, by (A 7), $\dddot{x} > p + 3(1-x^2)\dot{x}$. Since $1 - x^2$ and $\dot{x}$ are both negative, this means $\dddot{x}(t) > p$ as long as $\dddot{x}(t) > 0$ and $\tau \leqslant t < t_1$. Hence $\dddot{x}(t_1) \geqslant p$.

Noting that $\ddot{x}(t_1) = u_1(1-t_1^2)$, we have the first parts of (A 9) and (A 10), and by using (A 5) we obtain

$$1-t_1^2 < \frac{3\alpha+\alpha^2}{(1+\alpha)(2+\alpha)},$$

which completes the proof of (A 9) and (A 10).

Our aim is to prove that $\ddot{x}$, negative at t_1, remains negative until $x = 0$. If we show that $\ddot{x} < 0$ as long as $x \geqslant \sqrt{\frac{2}{3}}$, it will follow that $\dot{x} < -u_1$ during this time, and x will reach $\sqrt{\frac{2}{3}}$ (with $\dot{x} < 0$ and $\ddot{x} < 0$). After this, for as long as $\ddot{x} < 0$, we shall have $\ddot{x} < 0$ by (A 7) and consequently $\ddot{x} < 0$ until $x = 0$.

We show first that while $x^{\text{iv}}(t) < 0$ and $t > t_1$,

$$\ddot{x} < -\tfrac{1}{2}u_1t_1(3x-1). \tag{A 11}$$

Since $t_1 < 1$ and $1-t_1^2 < \frac{3}{2}\alpha$ by (A 9), and $u_1 > \alpha$ by (A 3), we have immediately from (A 8) that $x^{\text{iv}}(t_1) = u_1\{t_1(1-t_1^2)-6u_1\} < 0$. Suppose $x^{\text{iv}}(t) < 0$ for $t_1 \leqslant t \leqslant t_2$. Now $\ddot{x}(t)$ is negative at $t = t_1$, and as long as this persists we shall have

$$\frac{\ddot{x}(t)-\ddot{x}(t_1)}{x(t)-x(t_1)} = \frac{\dddot{x}(t^*)}{\dot{x}(t^*)} \quad \text{for some } t^* \text{ in} \quad (t_1, t).$$

Thus
$$\ddot{x}(t)-\ddot{x}(t_1) = \frac{1-x(t)}{-\dot{x}(t^*)}\dddot{x}(t^*) < \frac{1-x(t)}{u_1}\dddot{x}(t_1) \tag{A 12}$$

since $\dddot{x}$ decreases in $[t_1, t_2]$. Because $\dddot{x}(t_1)/u_1 < \frac{3}{2}\alpha < \frac{3}{2}u_1t_1$, from (A 10) and (A 3), and $\ddot{x}(t_1) = -u_1t_1$, (A 11) follows from (A 12).

Thus, for as long as $t_1 \leqslant t \leqslant t_2$ and $x \geqslant \sqrt{\frac{2}{3}}$, $\ddot{x}(t)$ is less than the negative constant $-\frac{1}{2}u_1t_1(\sqrt{6}-1)$; in these circumstances $\ddot{x}$ is never zero in $[t_1, t_2]$, and it follows that as long as $x \geqslant \sqrt{\frac{2}{3}}$ and $x^{\text{iv}}(t) < 0$ we have $\ddot{x}(t) < 0$.

Assembling the inequalities proved above, we have, for as long as $x \geqslant \sqrt{\frac{2}{3}}$ and $x^{\text{iv}}(t) < 0$,

$$\dddot{x} < \tfrac{3}{2}\alpha u_1, \quad \ddot{x} < \tfrac{1}{2}u_1t_1(1-3x), \quad t\alpha < (1+\alpha-x)\,t_1$$

(the last from (A 4)). Applying these to (A 8) we obtain

$$\begin{aligned} x^{\text{iv}}(t) &< \tfrac{3}{2}u_1t_1(1+\alpha-x)+\tfrac{1}{2}u_1t_1(1-3x)(1-x^2)-6\sqrt{\tfrac{2}{3}}\,u_1^2 \\ &= \tfrac{1}{2}u_1t_1(1-x)(2-3x^2-2x)+\tfrac{3}{2}u_1(\alpha t_1-4\sqrt{\tfrac{2}{3}}\,u_1). \end{aligned}$$

Since $\alpha t_1 < \alpha < u_1$, by (A 9) and (A 3), we deduce from this that, while $3x^2 \geqslant 2$ and $x^{\text{iv}}(t) < 0$, x^{iv} remains less than the negative constant $\frac{3}{2}u_1^2(1-4\sqrt{\frac{2}{3}})$. This means that $x^{\text{iv}}(t)$, and consequently $\ddot{x}(t)$, remain negative for as long as $x \geqslant \sqrt{\frac{2}{3}}$. This completes the proof.

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